Testing of component-based systems
Marc Aiguier, Bilal Kanso, Frédéric Boulanger, Christophe Gaston

To cite this version:
Marc Aiguier, Bilal Kanso, Frédéric Boulanger, Christophe Gaston. Testing of component-based systems. 19th Asia-Pacific Software Engineering Conference APSEC’12, Dec 2012, Hong-Kong, Hong Kong SAR China. pp.1-6, 2012. <hal-00782889>

HAL Id: hal-00782889
https://hal-ecp.archives-ouvertes.fr/hal-00782889
Submitted on 30 Jan 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Abstract—In this paper, we pursue our works on generic
modeling and conformance testing of component-based systems.
Here, we extend our theory of conformance testing to the testing
of component-based systems. We first show that testing a global
system can be done by testing its components thanks to the
projection of global behaviors onto local ones. Secondly, based
on our projection techniques, we define a framework to build
adequate test purposes automatically for testing components in
context of the global system where they are plugged in. The
underlying idea is to identify from any trace \( tr \) of the global
system, the trace of any component involved in \( tr \). Those projected
traces can be then seen as test cases that should be tested on
individual components.

Keywords: Component-based system, Conformance testing, Compo-
sitional testing, Testing in context, Projection, Test purpose.

INTRODUCTION

In the last decades, the component-based software ap-
proach [1], [2] has emerged due to the great advantages it
offers: modularity, re-usability, cost-effective solution. Com-
ponents are then designed, developed and validated in order to
be widely used, while complex software systems are described
recursively, at a higher level of abstraction, as interconnections
of those components. Hence, each sub-system (or component)
can be either a complex system itself or a simple component,
elementary enough to be handled without further decomposition.
Composition is used for fitting different components
together and then defining larger systems. Such a composition
is defined by operations which take components as well as the
nature of their interactions to provide a description of a new
and more complex component or system.

In [3], we proposed a formal framework for modeling basic
components viewed as abstract state-based systems. Compo-
nents were then modeled as coalgebras over \textit{sets}-endofunctor
with monads [4], [5] following Barbosa’s component defi-
nition [6], [7]. Monads enabled us to generically consider
a wide range of computation structures such as partiality,
non-determinism, etc. [5], and then to define components
independently of any computation structure. This definition
allowed us to unify in a same framework a large family
of state-based formalisms such as Mealy automata [9], [8],
Labeled Transition Systems [10], Input-Output Labeled Tran-
sition Systems [14], [11], etc. Larger systems are then built
by integrating components from integration operators defined
by composition of two basic ones: Cartesian product and
feedback. In [3], we showed that most standard integration
operators such as sequential and concurrent composition or
synchronous product are subsumed by our generic definition of
integration operators. Based on this framework, a conformance
testing theory has been defined in [3].

The ”plug and play” nature of component-based system
design leads naturally to build always bigger systems whose
correctness happens to be more and more difficult to assert.
This is of course due to the fact that analyzing big systems
generates state and time explosion problems, but it may also
be caused by the system architecture (e.g. distributed system)
which may complicate the ability to instrument the system in
order to observe behaviors to be analyzed. Even more, if a
”faulty” behavior is observed in such a system, the size of the
system is a problem to identify the cause of the fault at the
debugging phase.

All these reasons call to find ways to make system val-
idaion modular. Such methods enable to analyze a system,
subsystems per subsystems, in a modular way, rather than ”as
a whole”. Analyzed such systems are smaller (less prone to
generate explosion problems), more observable and control-
able (thus their behaviors are easier to cover), and debugging
is greatly facilitated.

Compositional testing [15], [20], [22] is viewed as one of
the most promising directions to bridge the gap between the
increasing complexity of systems and actual testing method
limits due to the reasons discussed above. Similarly to com-
positionality result in [20] establishing under certain hypothesis
that the conformance testing relation \( ioco \) is compositional
with respect to parallel composition and hiding, we have
established a compositionality result in [3]. This result expresses
that for the conformance relation \( ioco \)\(^1\) and \( n \) implementations
and specifications \( iut, \) and \( spec_i, 1 \leq i \leq n, \) each one modeled
by a component as defined in [3], if for each \( i, 1 \leq i \leq n, \)
\( iut, ioco spec_i, \) then for any integration operator of arity \( n \) (see
Definition 1.7), \( op(iut_1, \ldots, iut_n) ioco op(spec_1, \ldots, spec_n). \)
The compositionality result obtained in [3] is thus an extension
of Tretmans’s result [20] since it is established independently
of a given integration operator.

This result justifies the approach that consists in testing
separately the components of a system in order to build the

\(^1\)Actually, a slight extension of this relation to our components called \( cioco \)
in [3] (see Definition 2.1 in this paper).
Indeed each practice, such an integration theory is not enough. Such a result does not help to choose test purposes that are meaningful. Indeed each $\text{iut}_{t,i \leq n}$ is tested with respect to its specification $\text{spec}_{t,i \leq n}$, but since testing means selecting a finite number of executions (test cases) to evaluate the conformance, the question is then how to build a meaningful set of executions? Following approaches in [20] and [3] which are dedicated to model-based testing, we propose to extract test cases from specification. However, $\text{spec}_{t,i}$, standing alone, does not contain enough explanation to know how $\text{iut}_{t,i}$ will be used in the context of the whole system. This usage is in the end the only aspect that matters at test selection phases since all behaviors reflecting a non-conformance between $\text{iut}_{t,i}$ and $\text{spec}_{t,i}$ which are never activated in the context of the whole system. This last result is inspired from the approach proposed in [15], initially developed in the setting of IOSTS (symbolic automaton). In [15], only projection is defined, but no compositional result is given.

Based on this result, we will then propose a technique that strengthens testing of each component involved in a global system, by choosing suitable test purposes for them. This will be done by defining a projection mechanism that, from global behaviors of a system, will help generating test purposes capturing the behaviors of the sub-systems, that typically occur in the context of the whole system.

The paper is structured as follows: Section I recalls our framework for modeling components and systems. Section II introduces the conformance testing theory and discusses its main limitation for the validation of complex software systems. Section III presents the compositional result and shows how components can be tested while taking the system to which they belong into account.

I. COMPONENTS AND SYSTEMS

A. Components

In [3], a component is defined as a generalized Mealy automaton in which the dependence between outputs and both current state and inputs is relaxed from a strict deterministic, to encompass more complex behaviors such as partiality, non-determinism, etc. Components are defined using terminology and notations of coalgebras [24] and monads [4]. Hence, a component in [3] is a coalgebra $(S, \alpha)$ over a signature $\tau = \{\text{mark, plus, average, screen, nb}\}$. The monads have been introduced because they allow us to generically consider many computation situations such as determinism, non-determinism, partiality, etc. (see [3], [5] for more explanations).

Here, to make easier the readability of the paper, we restrict ourself to a particular case when $T$ stands for the powerset monad $\mathcal{P}$. The generalization to any monad $T$ does not raise any difficulties.

Definition 1.1 (Component): Let $I$ and $O$ be two sets denoting, respectively, the input and output domains. A component $C$ over $(I, O)$ is a triplet $(S, \text{init}, \alpha)$ where:

- $S$ is the set of states of $C$;
- $\text{init} \in S$ is a distinguished element denoting the initial state of $C$;
- $\alpha : S \times I \rightarrow \mathcal{P}(O \times S)$ is the transition function.

Example 1.1: To illustrate our approach, we will consider in this paper a simple system $S$ that computes grade averages presented in Figure 1. This system $S$ is built from two basic components: a “graphical interface” that helps the user to make various operations on grades and a “calculator” that receives operation commands from the user, performs the requested operation, and reports back to the user.

![Fig. 1: Grade averages system as a composition of the graphical interface and the calculator](image)

In our framework, the graphical interface is modeled as the component $G = (\{s_0, s_1, s_2, s_3, s_4, s_5\}, s_0, \alpha_1)$ over the
signature
\[ \Sigma_1 = (\{\text{mark, plus, average, nb, res}, +, /, \text{screen, val}\}) \]

and the calculator as the component \( C = (\{q_0, q_1, q_2, q_3\}, q_0, \alpha_2) \) over the signature
\[ \Sigma_2 = (\{+, -, 0, \}, \{\text{res, }\}) \]

\( \alpha_1 \) (resp. \( \alpha_2 \)) is depicted in the box at the top side (resp. bottom side) of Figure 1.

The semantics of a component is characterized by the set of finite sequences of couples (input/output), that is illustrated by the following definition:

**Definition 1.2 (Component finite traces):** The finite trace from a state \( s \) of a component \( C \), noted \( \text{Trace}_{\Sigma}(s) \), is the whole set of the finite input-output sequences \( \langle i_0|o_0, \ldots, i_n|o_n \rangle \) such that there exists a finite sequence \( \langle s_0, \ldots, s_{n+1} \rangle \in S^* \) of states where for every \( j, 0 \leq j \leq n, (o_j, s_{j+1}) \in \alpha(s_j)(i_j) \) with \( s_0 = s \).

Hence, the set of traces of \( C \), noted \( \text{Trace}(C) \), is the set \( \text{Trace}_{\Sigma} \text{=} \text{init} \).

In the following, we note \( \alpha(s)(i)_i \) (resp. \( \alpha(s)(i)_{ij} \)) the set composed of all first arguments (resp. second arguments) of couples in \( \alpha(s)(i) \).

**B. Systems**

Larger systems are built by composition from two basic operators: Cartesian product and feedback.

**Cartesian product:** The cartesian product is a composition where both components are executed simultaneously when triggered by a pair of input values.

**Definition 1.3 (Cartesian product \( \otimes \)):** Let \( C_1 = (S_1, \text{init}_1, \alpha_1) \) and \( C_2 = (S_2, \text{init}_2, \alpha_2) \) be two components over \((I_1, O_1)\) and \((I_2, O_2)\) respectively. \( C_1 \otimes C_2 \), the cartesian product of \( C_1 \) and \( C_2 \), is the component \( (S_1 \times S_2, \text{init}_1 \times \text{init}_2, \alpha) \) over \((I_1 \times I_2) \times (O_1 \times O_2)\) where \( \alpha \) is the mapping defined for every \((i_1, i_2) \in I_1 \times I_2\) and every \((s_1, s_2) \in S\) by:
\[ \alpha((s_1, s_2))(i_1, i_2) = \{((o_1, o_2), (s'_1, s'_2))|(o_k, s'_k) \in \alpha(s_k)(i_k) \text{ for } k = 1, 2\} \]

**Feedback:** The concept of feedback composition is intrinsic in dynamic system modeling in control theory [16, 17]. Here, we fit it to discrete systems. A component with feedback has directed cycles, where an output from a component is fed back to affect an input of the same component. That means the output of a component in any feedback composition depends on an input value that in turn depends on its own output value.

First, we introduce feedback interfaces for defining correspondences between outputs and inputs of components and only keeping both inputs and the outputs that are not involved in feedback.

**Definition 1.4 (Feedback interface):** A feedback interface over an interface signature \((I, O)\) is a triplet \( I = (f, \pi_1, \pi_o) \) where \( f : I \times O \rightarrow I \) is a mapping, and \( \pi_i : I \rightarrow I' \) and \( \pi_o : O \rightarrow O' \) are surjective mappings such that \( \forall (i, o) \in I \times O, f(f(i, o), o) = f(i, o) \) and \( \pi_i(i) = \pi_i(f(i, o)) \).

The mapping \( f \) specifies how components are linked and which parts of their interfaces are involved in the composition process. It finds the new value of the input that it is both a valid input and a valid output of the component, given its current state. Both mappings \( \pi_i \) and \( \pi_o \) can be thought as extensions of the hiding connective found in process calculi [19].

The feedback operator\(^2\) we consider here is synchronous. That means the reaction of a system takes no observable time [18] and its outputs are produced synchronously with its inputs. More precisely, at some reaction \( r \), the output of component \( C \) on \( r \) must be available to its inputs in the same reaction \( r \). The synchronous feedback requires then the existence of an instantaneous fix-point (i.e. defined at the same time and not deferred of one unit). This gives rise to the notion of well-formed feedback interface.

**Definition 1.5 (Well-formed feedback interface):** Let \( C \) be a component over \( \Sigma = (I, O) \) and \( \mathcal{L} = (f, \pi_1, \pi_o) \) be a feedback interface over \( \Sigma \). We say that \( \mathcal{L} \) is well-formed w.r.t \( C \) if, and only if for every state \( s \in S \) and every sequence of inputs \( x_1, \ldots, x_n \), there exists a sequence of outputs \( y_1, \ldots, y_n \) such that for every \( j, 1 \leq j < n, y_j \in \alpha(s)(f(x_j, y_j)) \).

We want to build a component that hides the feedback of a component \( C \). As one can see in Figure 2, the feedback component \( \otimes_{\mathcal{L}}(C) \) is defined over the signature \((I', O')\). The outputs are then hidden from any state \( s \) that are fed back as inputs to \( s \). The result is a component with input and output sets \( I' \) and \( O' \) respectively. This is done by means of the feedback interface \( \mathcal{L} = (f, \pi_1, \pi_o) \). Let us suppose that the current state of \( C \) at the \( n^{th} \) reaction is \( s_n \in S \) and the current external input is \( x(n) \in I \); then let us compute both new input \( x'(n) \in I' \) and output \( y'(n) \in O' \) when \( C \) is triggered by \( x(n) \).

First, by \( f \), we compute the input \( \bar{x}(n) = f(x(n), y(n)) \). Then, \( \bar{x}(n) \) becomes the new input of \( C \). Indeed, component \( C \) reacts by updating its state to \( s_{n+1} \) and producing an output \( y(n) \) (such a \( y(n) \) exists since \( \mathcal{L} \) is well-formed w.r.t \( C \)). Second, by means of \( \pi_1 \) and \( \pi_o \), we hide both input and output involved in the feedback, and then produce the input \( x'(n) = \pi_1(\bar{x}(n)) \) and the output \( y'(n) = \pi_o(y(n)) \) of the feedback component \( \otimes_{\mathcal{L}}(C) \).

\(^2\)There is another kind of feedback called relaxed feedback. Interested readers may refer to [3].
Definition 1.6 (Synchronous feedback \( \otimes \)): Let
\[ I = (f, \pi_i, \pi_o) \] be a feedback interface over \( \Sigma = (I, O) \). Let \( C = (S, \text{init}, \alpha) \) be a component over \( \Sigma \) such that \( I \) is well-formed w.r.t. \( C \). \( \otimes_I(C) \), the synchronous feedback over \( I \), is the component \( C' = (S, \text{init}, \alpha') \) over \( \Sigma' = (I', O') \) where \( \alpha' \) the mapping defined for every \( s \in S \) and every \( i' \in I' \) by: \( \alpha'(s)(i') = \{ (a', s') \mid \exists (i, o) \in (I \times O), (s', o') \in \alpha(s)(f(i, o)), \pi_i(i) = i' \text{ and } \pi_o(o) = o' \} \)

Complex operators and systems:
As previously explained, from Cartesian product and feedback operators, we can build more complex ones by composition.

Definition 1.7 (Complex operator): The set of complex operators, is inductively defined as follows:

- \( \_ \) is a complex operator of arity 1;
- if \( op_1 \) and \( op_2 \) are complex operators of arity \( n_1 \) and \( n_2 \) respectively, then \( op_1 \otimes op_2 \) is a complex operator of arity \( n_1 + n_2 \);
- if \( op \) is complex operator of arity \( n \) and \( I \) is a feedback interface, then \( \otimes_I(op) \) is a complex operator of arity \( n \).

In Example 1.2, as an example of a complex operator, we show how the sequential operator can be defined in our framework.

Example 1.2: The sequential composition \( \triangleright \) of two components \( C_1 \) and \( C_2 \) corresponds to a composition where both components \( C_1 \) and \( C_2 \) are interconnected side-by-side and the output of one is the input of the other. This kind of composition can be naturally defined in our framework as follows:

\[ \triangleright((C_1, C_2)) = \otimes_I((C_1 \otimes C_2)) \]

where \( I = (f, \pi_i, \pi_o) \) is the feedback interface defined \( \forall(i, i') \in I_1 \times I_2, \forall(o, o') \in O_1 \times O_2 \) by:

\[ f((i, i'), (o, o')) = (i, o), \pi_i((i, i')) = i \text{ and } \pi_o((o, o')) = o' \]

Other standard operators have been also defined similarly in [3]

Complex operators will not be necessarily defined when applied to a sequence of components. Indeed, for a complex operator of the form \( \otimes_I(op) \), according to the component \( C \) resulting from the evaluation of \( op \), the interface \( I \) has to be defined over the signature of \( C \) and the feedback over \( C \) has to be well-formed. Hence, a system will be the component resulting from the evaluation of complex operators, from a sequence of components, when it is defined.

Definition 1.8 (Systems): Let \( \mathcal{C} \) be a set of components. The set of systems over \( \mathcal{C} \) is inductively defined as follows:

- for any \( C \in \mathcal{C} \), a component over a signature \( \Sigma, (_C(C) = C \) is a system over the signature \( \Sigma \) and \( C \) is defined for \( C \);
- if \( op_1 \otimes op_2 \) is a complex operator of arity \( n = n_1 + n_2 \) then for every sequence \( (C_1, C_2, \ldots, C_n) \) of components in \( \mathcal{C} \) with each \( C_i \) over \( \Sigma_i = (I_i, O_i) \), if both \( op_1 \) and \( op_2 \) are defined for \( C_1, C_2, \ldots, C_n \) and \( C_{n+1}, \ldots, C_n \) respectively, then \( op_1 \otimes op_2 \) is a system over \( \Sigma = \left( \prod_{i=1}^{n} I_i, \prod_{i=1}^{n} O_i \right) \) and \( op_1 \otimes op_2 \) is defined for \( (C_1, \ldots, C_n) \), else \( op_1 \otimes op_2 \) is undefined for \( (C_1, \ldots, C_n) \);
- if \( C \) is a feedback interface over \( \Sigma \) and \( I \) is well-formed w.r.t. \( \Sigma \), then \( \otimes_I(op)(C_1, \ldots, C_n) = C \) is a system over \( \Sigma' \) and \( \otimes_I(op)(C_1, \ldots, C_n) = \otimes_C(S) \) is a system over \( \Sigma' \) and \( \otimes_I(op)(C_1, \ldots, C_n) \) is undefined for \( (C_1, \ldots, C_n) \).

We introduce the definition of a sub-system involved in a given system. This intuitively allows us to characterize the set of all basic sub-systems from which the global system can be built.

Definition 1.9 (Sub-systems): Let \( S = op(C_1, \ldots, C_n) \) be a system over a signature \( \Sigma \). The set of sub-systems of \( S \), noted \( \text{Sub}(S) \), is inductively defined on the structure of \( op \) as follows:

- if \( op = \_ \) then \( \text{Sub}(S) = \{ S \} \);
- if \( op = op_1 \otimes op_2 \) with \( op_1 \) and \( op_2 \) of arity \( n_1 \) and \( n_2 \) respectively (i.e. \( n = n_1 + n_2 \)), then \( \text{Sub}(S) = \{ S \} \cup \text{Sub}(op_1(C_1, \ldots, C_n)) \cup \text{Sub}(op_2(C_{n+1}, \ldots, C_n)) \);
- if \( op = \otimes_I(op') \), then \( \text{Sub}(S) = \{ S \} \cup \text{Sub}(op'(C_1, \ldots, C_n)) \).

Example 1.3: The system \( S \) to compute grade averages is obtained as a composition of \( \mathcal{G} \) and \( \mathcal{C} \) using our basic integration operators. Hence to define the system \( S \), we first apply the Cartesian product \( \otimes((\mathcal{G}, \mathcal{C})) \) to \( \mathcal{G} \) and \( \mathcal{C} \) over the signature \( \Sigma_{\otimes} = (I_{\otimes}, O_{\otimes}) \) with: \( I_{\otimes} = \{ \text{mark, plus, average, nb} \} \times \{ \text{val, +, /} \} \) and \( O_{\otimes} = \{ \text{val, screen, } \bot \} \times \{ \text{val, res} \} \). We can then see that:

- both outputs + and / of \( \mathcal{G} \) are returned as inputs of \( \mathcal{C} \);
- the output "res" of \( \mathcal{C} \) is returned as input of \( \mathcal{G} \).

Then, we apply the synchronous feedback to \( \otimes((\mathcal{G}, \mathcal{C})) \). This leads to the operator \( \otimes_I \) over the interface signature \( I = (f, \pi_i, \pi_o) \) as follows:

\[ f: \quad I_{\otimes} \times O_{\otimes} \rightarrow I_{\otimes} \]

\[ (((i, i'), (o, o')) \rightarrow \begin{cases} (i, o) \quad \text{if } i' = o \\ (i, i') \quad \text{otherwise} \end{cases} \]

\[ \pi_i: \quad I_{\otimes} \rightarrow I_{\mathcal{G}} \cup I_{\mathcal{C}} \]

\[ (i, i') \rightarrow \begin{cases} i \quad \text{if } i' \in O_{\mathcal{C}} \\ i' \quad \text{otherwise} \end{cases} \]

\[ \pi_o: \quad O_{\otimes} \rightarrow O_{\mathcal{G}} \cup O_{\mathcal{C}} \]

\[ (o, o') \rightarrow \begin{cases} o' \quad \text{if } o \in I_{\mathcal{G}} \\ o \quad \text{otherwise} \end{cases} \]

\( \otimes' \) is the signature of the synchronous feedback.
Applying $\diamond_X$ to $\otimes((G, C))$ leads to a new component $\diamond_X ((G, C))$ (see Figure 3) where all outputs of $G$ (i.e., $+$, $/$ and val) that are fed back to $C$ and the output “res” of $G$ that is fed back to $G$ are hidden (i.e., synchronized).

Fig. 3: Component $\diamond_X ((G, C))$

II. CONFORMANCE TESTING

Conformance testing theory is usually based on the comparison between the behavior of a specification and an implementation using a conformance relation. The goal of this relation is to specify what the conformance of an implementation is with respect to its specification. It has been shown that the input-output conformance relation $cioco$ is the most suitable for testing our components [3]. This relation distinguishes input and outputs actions, and requires that the implementation behaves according to a specification, but also allows behaviors on which the specification puts no constraint.

The specification $spec$ of a component is the formal description of its behavior given by a component over a signature $(I, O)$. On the contrary, its implementation $iut$ is an executable component, which is considered as a black box [25], [26]. We interact with the implementation through its interface, by providing inputs to stimulate it and observing its behavior through its outputs. Hence, to be able to treat the implementation $iut$, we make the following two assumptions about it:

- The implementation $iut$ can be modeled as a component $(S, init, \alpha)$ over the signature $(I', O')$ with $I \subseteq I'$ to allow the implementation to accept all the inputs of the specification and $O' \subseteq O$ to allow the specification to accept all the responses of the implementation.
- $iut$ is input-enabled, i.e., at any state, it must produce answers for all inputs provided by the environment: $\forall (s, i) \in S \times I, \exists (o, s') \in O \times S$ such that $(o, s') \in \alpha(s)(i)$

The conformance relation that we will call here $cioco$ is a slight adaptation of the standard relation $ioco$ [11].

Definition 2.1: (cioco) Let $spec$, $iut$ be two components over $(I, O)$ and $(I', O')$ respectively such that $I \subseteq I'$, $O' \subseteq O$ and $iut$ is input-enabled. $iut$ is in conformance with $spec$, noted $iut$ $cioco$ $spec$, if and only if

\[ \forall tr \in Trace(spec), \forall i \in I, \\
Out(iut \text{ after } (tr, i)) \subseteq Out(spec \text{ after } (tr, i)) \]

where for any component $C$, any finite trace $tr$, and any input $i$ of $C$, $Out(C \text{ after } (tr, i))$ is the set

\[ \{o | tr.(i|o) \in Trace(C)\} \]

When the $Out(spec \text{ after } (tr, i))$ is empty, that ensures the quiescence notion introduced by Tretmans in [13].

Similarly to [20], we studied in [3] compositionality properties for $cioco$ over integration operators defined in Section I-B. We then proved the following theorem:

Theorem 2.1 (Compositionality [3]): Let $op$ be a complex operator of arity $n$. Let $iut_1, \ldots, iut_n, spec_1, \ldots, spec_n$ be input-enabled components such that $\forall i, 1 \leq i \leq n$, $iut_i$ $cioco$ $spec_i$, then one has $op(iut_1, \ldots, iut_n)$ $cioco$ $op(spec_1, \ldots, spec_n)$.

That means if single components of a system conform to their specifications, the whole system built over our integration operators is in accordance with its specification, unless the specification model is input-enabled. Such a testing compositionality result theory provides a way to test the integrated system only by testing its sub-systems i.e., there is no need to re-test its conformance correction. Hence, once this property is verified, the correctness of the integrated system is obtained from the correctness of the individual components. To test the integrated system, it is not necessary to consider it as a whole, but it is enough to consider its sub-systems and test them separately. Indeed, the contraposition of this property is the following:

\[ \neg (op(iut_1, \ldots, iut_n) \text{ } cioco \text{ } op(spec_1, \ldots, spec_n)) \implies \exists i, 1 \leq i \leq n, \neg (iut_i \text{ } cioco \text{ } spec_i) \]

Thus, by looking at this new property, we can easily see that non-correctness of the integrated system under test $op(iut_1, \ldots, iut_n)$ implies that at least one of its components $iut_1, \ldots, iut_n$ is incorrect. In other words, that means to test $op(iut_1, \ldots, iut_n)$, it suffices to test $iut_1, \ldots, iut_n$ in isolation.

In the sequel, we will show how to improve significantly the result obtained in Theorem 2.1 by taking into account the global system in which components are plug in. This will be achieved by using projection mechanisms.

III. PROJECTION AND TEST PURPOSES

A. Projection and compositionality

Projection techniques [15] are defined by pruning from any global behavior $p$, all that does not concern the sub-system that we want to test. This will allow us to generate more relevant unit test cases to test individual components. As an illustration, let us again consider the system that computes grade averages (see Example 1.3). According to the result obtained in Theorem 2.1, to test the grade average system, it suffices to test separately the calculator $C$ and the controller $G$. Now, testing the calculator $C$ separately may lead to the consideration of test cases involving arithmetic operations which are irrelevant to computing student grade averages such
as subtraction or multiplication. This may cause test cases of interest to the system to be missed, i.e. test cases only bringing into play addition and division for grades ranging from 0 to 20. In the approach we propose in the following, we intend to generate a test purpose that guides the test derivation process of $C$ by only testing operations needed to compute grade averages. We do this by making a projection of this behavior on calculator component $C$.

**Definition 3.1 (Projection):** Let $S = \text{op}(C_1, \ldots, C_n)$ be a system over $(I, O)$. Let $\text{sub} \subseteq \text{Sub}(S)$ be a sub-system of $S$ over $(I', O')$. Let $tr = \langle i_1, o_1, \ldots, i_m, o_m \rangle \in \text{Trace}(S)$. The projection of $tr$ on $\text{sub}$, denoted by $tr_{\text{sub}}$, is the subset of $\text{Trace}(\text{sub})$ inductively defined as follows:

- if $\text{op} = -$, then $tr_{\text{sub}} = \{\{\text{tr}\}\}$;
- if $\text{op} = \text{op}_1 \otimes \text{op}_2$ with $\text{op}_1$ and $\text{op}_2$ of arity $n_1$ and $n_2$ respectively (i.e. $n = n_1 + n_2$), then$^5$

$$tr_{\text{sub}} = \begin{cases} \text{is the projection of } \langle i_{n_1}, o_{n_1}, \ldots, i_{n_2}, o_{n_2} \rangle \text{ on sub} & \text{if sub } \subseteq \text{Sub}(\text{op}_1(C_1, \ldots, C_n)) \\ \text{is the projection of } \langle i_{n_1}, o_{n_1}, \ldots, i_{n_2}, o_{n_2} \rangle \text{ on sub otherwise} & \end{cases}$$

- if $\text{op} = \text{op}_i$ with $\mathcal{I} = (f, \pi_s, \pi_o)$, then $tr_{\text{sub}} = \bigcup_{\text{sub} \subseteq \text{Iut}} tr_{\text{sub}}$ where

$$\begin{array}{l}
S' = \text{op}_i(C_1, \ldots, C_n) \\
\text{and } tr_{\text{sub}} = \{\langle i_1', o_1', \ldots, i_m', o_m' \rangle \mid \forall j, 1 \leq j \leq m, \exists o_j \in S' \\
\exists t_j \in S' \ni \forall j, 1 \leq j \leq m, \exists s_j \in S', \exists a_j \in \alpha^j \text{ with } i_j = \pi_i (i'_j) \text{ and } o_j = \pi_o (o'_j)\} \\
\end{array}$$

We then introduce the projection of a system on a one of its sub-systems.

**Definition 3.2 (Component in context):** Let $S$ be a system over $(I, O)$ and $\text{sub} \subseteq \text{Sub}(S)$ be a subsystem of $S$ over $(I', O')$. The component obtained by projecting $S$ on $\text{sub}$, noted $S_{\text{sub}}$, is the triple $(S, s^0, \alpha)$ defined by:

- $s^0 = ()$
- $S$ is the whole set of finite traces defined as follows:
  - $s^0 = ()$
  - $\{i, j \leq n, s^j = \{tr'.(i|o) \mid \exists t \in s^{j-1}, \exists i \in I', \exists o \in O, \exists tr \in \text{Trace}(S) \text{ such that } tr'.(i|o) \in tr_{\text{sub}}\}$

Hence, $S = \bigcup_{0 \leq j \leq \omega} s^j$

- $\alpha : S \times I' \rightarrow P(O' \times S)$ is the mapping which for every $\langle i_0, o_0, \ldots, i_m, o_m \rangle \in S$ and every input $i \in I'$ associates the set:

$$\Pi = \{\langle o, i, i_0, o_0, \ldots, i_m, o_m, i|o \rangle \mid \exists o \in O', \exists tr \in \text{Trace}(S) \text{ such that } \langle i_0, o_0, \ldots, i_m, o_m, i|o \rangle \in tr_{\text{sub}}\}$$

It is easy to see that the traces of the component $S_{\text{sub}}$ obtained by projection is a subset of the traces of the component $S$ itself.

**Example 3.1:** Consider again the grade average system $\text{op}_2(\text{op}_1(G, C))$ given in Figure 3. The projection $\text{op}_2(\text{op}_1(G, C))_{\text{sub}}$ of $\text{op}_2(\text{op}_1(G, C))$ on the calculator $C$ is given in Figure 4. By applying Definition 3.2, we only retain the $C$’s behaviors that are involved in the final behavior of $\text{op}_2(\text{op}_1(G, C))$. Only the addition and the division operations are specified in $\text{op}_2(\text{op}_1(G, C))_{\text{sub}}$, the specifications of both subtraction and multiplication operations are omitted due to their absence in the global system $\text{op}_2(\text{op}_1(G, C))$.

![Fig. 4: The projection $\text{op}_2(\text{op}_1(G, C))_{\text{sub}}$ of $\text{op}_2(\text{op}_1(G, C))$ on the calculator $C$](image)

Such projected traces will be the cornerstone to improve the compositionality result presented in Theorem 2.1 and to define test purposes dedicated to test components separately while taking into account the behavior of the global system.

**Theorem 3.1 (Compositionality with projection):** Let $\text{op}$ be a complex operator of arity $n$. Let $\text{Iut}_1, \ldots, \text{Iut}_n$ be input-enabled implementations and $\text{spec}_1, \ldots, \text{spec}_n$ their specifications respectively. Then, one has $\forall i, 1 \leq i \leq n$

$$(\text{Iut}_1 \text{ cioco } \text{op}(\text{spec}_1, \ldots, \text{spec}_n)_{i_{\text{spec}_1}}), \ldots, (\text{Iut}_n \text{ cioco } \text{op}(\text{spec}_1, \ldots, \text{spec}_n)_{i_{\text{spec}_n}})$$

$$\Longrightarrow \text{op}(\text{Iut}_1, \ldots, \text{Iut}_n) \text{ cioco } \text{op}(\text{spec}_1, \ldots, \text{spec}_n)$$

**Proof:** Sketch of the proof

This is proven by structural induction on the integration operator $\text{op}$. The main difficulty is to prove the property preservation over both Cartesian product and feedback operator. Then, we need the following two theorems:

**Theorem 3.2 (Compositionality for Cartesian product):** Let $C_1$ and $C_1'$ be two components over $(I_1, O_1)$, and $C_2$ and $C_2'$ be two components over $(I_2, O_2)$. Then, we have:

$$C_1 \text{ cioco } \otimes ((C_1', C_2'))_{i_{\text{spec}_1'}} \quad C_2 \text{ cioco } \otimes ((C_1', C_2'))_{i_{\text{spec}_2'}}$$

$$\Longrightarrow \otimes ((C_1, C_2)) \text{ cioco } \otimes ((C_1', C_2'))$$

**Theorem 3.3 (Compositionality for feedback operator):**

Let $\Sigma = (I, O)$ be a signature and $\mathcal{I} = (f, \pi_s, \pi_o)$ be a feedback interface. Let $C_1 = (s_1, \alpha_1)$ and $C_2 = (s_2, \alpha_2)$ be two components over $\Sigma$. Then, we have:

$$C_1 \text{ cioco } \otimes (C_2)_{i_{\text{spec}_2}} \Longrightarrow \text{op}(C_1, C_2)$$
The proof of both theorems 3.2 and Theorem 3.3 is given in Appendix.

Theorem 3.1 then provides a way to test the integrated system only by testing the projection of that system on its sub-systems. As a consequence, to test the integrated system, it is not necessary to consider it as a whole, but it is enough to consider the projection of that system on its sub-systems (which may be done at different development steps and eventually developed by different teams) and test them separately.

Comparing this result with our previous result presented in [3] or Tretmans’s result [20], the new result does not require that the specifications are input-enabled. This last property is often hard to get in practice due to the fact that system input domains are usually too large.

B. Test purpose

A specification model usually contains a growth of exponential states which makes the testing process difficult even impossible to be implemented. To cope with this problem, test purposes can be used. A test purpose is a description of the part of the specification that we want to test and for which test cases are later generated. In [14], they are described independently from the specification by construction. In order to guide the test derivation process in our approach, we have preferred, as in [23], to describe test purposes by selecting the part of the specification that we want to explore. We therefore consider a test purpose as a tagged finite computation (FCT) tree of the specification. The leaves of the FCT which correspond to paths that we want to test are tagged accept. All internal nodes on such paths are tagged skip, and all other nodes are tagged ⊙. Formally, FCT is defined as follows:

Definition 3.3 (Finite computation tree of component): Let $(S, s_0, α)$ be a component over $(I, O)$. The finite computation tree of depth $n$ of $C$, noted $FCT(C, n)$, is the triplet $(S_{FCT}, s_{FCT}^0, α_{FCT})$ defined by:

- $S_{FCT}$ is the whole set of $C$-paths. A $C$-path is defined by two finite sequences of states and inputs $(s_0, . . . , s_n)$ and $(i_0, . . . , i_{n-1})$ such that:

  $\forall j, 1 \leq j \leq n, s_j \in α(s_{j-1})|i_{j-1}|_2$

- $s_{FCT}^0$ is the initial $C$-path $(s_0, (i_0))$

- $α_{FCT}$ is the mapping which for every $C$-path $\langle (s_0, . . . , s_n), (i_0, . . . , i_{n-1}) \rangle$ and every input $i \in I$ associates the set:

  $Γ = \{ (α, (s_0, . . . , s_n, s'), (i_0, . . . , i_{n-1}, i)) \mid (α, s') \in α(s_n)(i) \}$

In this definition, $S_{FCT}$ is the set of the nodes of the tree and $s_{FCT}^0$ its root. Each node is represented by the unique $C$-path $\langle (i_0, . . . , i_{n-1}) \rangle$ which leads to it from the root. $α_{FCT}$ gives, for each node $p$ and for each input $i$, the set of nodes $Γ$ that can be reached from $p$ when the input $i$ is submitted to $C$.

We intend in the following to extend the notion of test purpose proposed in [3] to test purpose in context. This latter allows us to test, from a global behavior of a system, the behavior of its involved sub-systems and then guide the component testing intelligently by taking into account the way components are used in systems. Thus, taking a behavior $p$ of a system $S$, we intend to define test purposes that are able to test the behavior $p_i$ of each sub-system $S_i ∈ ⊰sub(S)$. We identify therefore for each sub-system all its finite paths that are involved in constructing the whole behavior of $S$.

Definition 3.4 (Test purpose in context): Let $S$ be a system over $(I, O)$. Let $sub ∈ ⊰sub(S)$ be a sub-system of $S$ and $sub' = S_{sub}$, the projection of $S$ on $sub$. Let $FCT(sub, n) = (S, s_0, α)$ be the finite computation tree of $sub$. A test purpose in context $TP$ for $sub$ is a mapping $TP : S_{FCT} → \{accept, skip, ⊙\}$ such that:

- for every node $p = ⟨i_0|θ_0, . . . , i_m|θ_m⟩ ∈ Trace(sub')$, $TP(p) = accept$;
- if $TP(⟨i_0|θ_0, . . . , i_m|θ_m⟩) = accept$, then:
  $\forall j, 0 ≤ j ≤ m, TP(⟨i_0|θ_0, . . . , i_{j-1}|θ_{j-1}⟩) = skip$
- $TP(⟨⟩) = skip$
- if $TP(⟨i_0|θ_0, . . . , i_k|θ_k⟩) = ⊙$, then:
  $TP(⟨i_0|θ_0, . . . , i_k|θ_k⟩, ⟨i_{k+1}′|θ_{k+1}′⟩, . . . , i_l|θ_l⟩) = ⊙$
  for all $k < k' ≤ n$ and for all $(i_j′)_{k ≤ l < n} ∈ I'$ and $(i_j''|θ_j'')_{k ≤ l < n} ∈ O'$.

In order to build a test purpose for a subsystem $sub$, we identify all finite paths of its finite computation tree FCT whose traces embody traces in $Trace(sub')$ and we tag them with accept. We then tag every node which represents a prefix of an accepted behavior with skip. The other nodes, which lead to behaviors that we do not want to test, are tagged with ⊙.

Example 3.2: In this example, we intend to build a test purpose dedicated to test the behavior of the calculator component $C$ in the context of the system computing grade averages. To do so, we first build the finite computation tree $FCT(C, 4)$ of $C$ that we present in Figure 5. Second, each state of $FCT(C, 4)$ reachable after each trace $tr$ of the projection $⊙_T ⊲(G, C)$ on $C$ (see Figure 3) is tagged with accept. Then, $p_3$ and $p_{11}$ are only tagged with accept. All nodes leading from the root $init$ to $p_9$ or $p_{11}$ are tagged with skip (i.e. $p_1, p_3, p_5$ and $p_7$). Finally, all other states are tagged with ⊙.

Thus, testing of $C$ is re-enforced as far as student grade averages computing is concerned: only behaviors related to grade average computing are chosen and then the behaviors of $C$ that are not activated in the global system $⊙_T ⊲(G, C)$ are not tested. This allows us to restrict the test domain to the one under consideration.

Finally, we use the algorithm developed in Algorithm 1 to generate correct and sound test cases. Given an implementation iut of a subsystem $sub$ of a system $S$ and the test purpose
input : a test purpose
\( TP : FCT = (S, s_0, \alpha) \rightarrow \{\text{accept, skip, } \odot\} \)
and an implementation iut
output: a test case \([i_0|o_0, i_1|o_1, \ldots, i_n|o_n, \text{ verdict}]\)

**Preliminaries:**
\( \text{Next}(CS, i|o) \) returns the set of directly reachable states from the current set of states \( CS \) after executing \( i|o \);
\( \text{NextSkip}(CS, i|o) \) returns the set of states in \( \text{Next}(CS, i|o) \) which are labeled by skip;
\( \text{NextPass}(CS, i|o) \) returns the set of states in \( \text{Next}(CS, i|o) \) which are labeled by accept;

**Initialization:**
\( i \leftarrow \text{ChooseInputFrom}(\{i \mid \alpha(s)(i) \text{ is defined}\}); \)
\( o \leftarrow \text{ReactionOf}(iut, i); \)
\( CS \leftarrow \{s\} \text{ / set of explored states}; \)
\( TC \leftarrow [] \text{ / initialization of the test case}; \)
\( \text{// sending stimuli to iut and waiting for its output as long as a verdict is not reached} \)

\[
\text{while } \text{NextSkip}(CS, i|o) \neq \emptyset \text{ and NextPass}(CS, i|o) = \emptyset \text{ do}
\]
\( TC \leftarrow \text{Concatenate}(TC, i|o); \)
\( CS \leftarrow \text{Next}(CS, i|o) \)
\( i \leftarrow \text{ChooseInputFrom}(\{i \mid i \in s \in CS \mid \alpha(s)(i) \text{ is defined}\}); \)
\( o \leftarrow \text{ReactionOf}(iut, i); \)
\( TC \leftarrow \text{Concatenate}(TC, i|o) ;\)
\( \text{// the emission from the iut is not expected with regards to the specification} \)
\[
\text{if } \text{Next}(CS, i|o) = \emptyset \text{ then}
\]
\( TC \leftarrow \text{Concatenate}(TC, \text{FAIL}); \)
\( \text{// the emission from the iut is specified, but not compatible with the test purpose} \)
\[
\text{if } \text{Next}(CS, i|o) \neq \emptyset \text{ and NextSkip}(CS, i|o) = \emptyset \text{ then}
\]
\( TC \leftarrow \text{Concatenate}(TC, \text{INCONC}); \)
\( \text{// all next states directly reachable from the set of current set are accept ones} \)
\[
\text{if } \text{Next}(CS, i|o) = \text{NextPass}(CS, i|o) \text{ and NextPass}(CS, i|o) \neq \emptyset \text{ then}
\]
\( TC \leftarrow \text{Concatenate}(TC, \text{PASS}); \)
\( \text{// some of the next states are labeled by accept, but not all of them} \)
\[
\text{if } \text{NextPass}(CS, i|o) \subset \text{Next}(CS, i|o) \text{ and NextPass}(CS, i|o) \neq \emptyset \text{ then}
\]
\( TC \leftarrow \text{Concatenate}(TC, \text{WeakPASS}); \)
\( \text{return } TC; \)

**Algorithm 1:** Test generation algorithm

**IV. Conclusion**

This paper extends our previous work [3] which defines a generic testing conformance theory. We have proposed an approach to test components that are typically involved in the whole system by defining test purposes from the global behaviour of the whole system. Such test purposes are given in a accurate way by defining a projection mechanism taking a global behaviour \( p \) of the whole system and keeping only the part of \( p \) being activated in the sub-system that we want to
Thus, our method for generating test purposes from the global system specification helps to generate relevant unit test cases to test individual components.

June 11, 2012

V. APPENDIX

Proof: Compositionality for synchronous feedback (Theorem 3.3)

We first need to prove the following lemma:

Lemma 5.1: Consider two components C1 and C2, then we have: C1 cioco ⊗T (C2) |i,c2 ⟷ ∀tr ∈ Trace(⊗T (C1)) ∩ Trace(⊗T (C2)), tr |i,c2 ⊆ tr |i,c1 .

Proof: Let us prove this point by induction on the structure of a trace tr in Trace(⊗T (C1)) ∩ Trace(⊗T (C2)). Let 

- Basic Step: tr = {} is empty trace. 
tr |i,c2 = {} ⊆ tr |i,c1 trivially holds.

- Induction Step: Let us write tr as concatenation of two finite traces: tr = ⟨i1 |o1, i2 |o2, . . . , in−1 |on−1, i n |on⟩. Let σ = ⟨i 1′| o 1′, . . . , i n−1′|o n−1′, i n′|o n′⟩ ∈ tr |i,c2 and let us prove that σ ∈ tr |i,c1 . σ ∈ tr |i,c2 then according to the definition of tr |i,c2 , there exists a finite sequence of states s0, . . . , sn of S2 such that ∀1 ≤ j ≤ n:

\[ o_j′, s_j ∈ α(\langle i_j′, o_j′ \rangle) \]

and

\[ π(i_j′) = i_j \quad \text{and} \quad π(o_j′) = o_j \]

Now, by induction hypothesis, we have σ = ⟨i 1′| o 1′, . . . , i n−1′|o n−1′|on−1⟩ ∈ tr |i,c1 , then according to the definition of tr |i,c1 , there exists a finite sequence of states s 0, . . . , s n−1 of S1 such that ∀1 ≤ j ≤ n − 1:

\[ o_j′, s_j′ ∈ α(\langle i_j′, o_j′ \rangle) \]

and

\[ π(i_j′) = i_j \quad \text{and} \quad π(o_j′) = o_j \]

One has that σ = ⟨i 1′| o 1′, . . . , i n−1′|o n−1′|on−1⟩ ∈ Trace(⊗T (C1))∩ Trace(⊗T (C2)) since σ ∈ tr |i,c2 and tr |i,c2 ∈ Trace(⊗T (C2)) (see Definition 3.2). That means that o n′ ∈ Out(⊗T (C2)) |i,c2 after ⟨⟨i 1′| o 1′, . . . , i n−1′|o n−1′, i n′⟩⟩

But we know that C 1 is input-enabled, then i n′ is inevitably a valid of the state s n−1. Hence, one has

o n′ ∈ Out(C 1 after ⟨⟨i 1′| o 1′, . . . , i n−1′|o n−1′, i n′⟩⟩ because of C 1 cioco ⊗T (C 2) and that means there exists s n′ ∈ S 1 such that (o n′, s n′) ∈ α1(⟨i n′| o n′⟩) since C 1 is well-formed for L. We know also π(i n′) = i n and π(o_n′) = o_n, thus ⟨i 1′| o 1′, . . . , i n−1′|o n−1′, i n′⟩ ∈ Tra_c 1. Consequently, tr |i,c2 ⊆ tr |i,c1 .

Let us now prove Theorem 3.3. Let C 1 and C 2 be two components over (I, O), and ⊗T (C 1) and ⊗T (C 2) over (I′, O′).

Let tr = ⟨i 1| o 1, . . . , i n|o n⟩ ∈ Trace(⊗T (C 1)) ∩ Trace(⊗T (C 2)) and (i n+1, o n+1) ∈ I′ × O′ such that:

o n+1 ∈ Out(⊗T (C 1) after (tr, i n+1))

Then, let us prove that

o n+1 ∈ Out(⊗T (C 2) after (tr, i n+1))

Let us define the set X by

References


$$\{i'_{n+1} \mid \langle i'_1, i'_2, \ldots, i'_n | o'_n, i'_{n+1} | o'_{n+1} \rangle \in tr. < i_{n+1} | o_{n+1} >_c \}$$
of all inputs enabling in $C_1$ after projecting the trace $tr. \langle i_{n+1} | o_{n+1} \rangle$ on $C_1$. Since, $tr \in_c C_2 \subseteq tr \in_c (Y)$ (By Lemma 5.1), we can extract from $X$ the set:

$$Y = \{i'_{n+1} \mid \langle i'_1, i'_2, \ldots, i'_n | o'_n, i'_{n+1} | o'_{n+1} \rangle \in tr \in_c C_1 \text{ and } \langle i'_1, i'_2, \ldots, i'_n | o'_n \rangle \in tr \in_c C_2 \}$$
of all inputs enabling in $C_1$ after the traces obtained by projecting $tr$ on $C_2$.

In the same manner, let us define the set

$$Z = \{i'_{n+1} \mid \langle i'_1, i'_2, \ldots, i'_n | o'_n, o'_{n+1} \rangle \in tr. < i_{n+1} | o >_c \text{ and } o \in Out(\otimes C_2) \text{ after } (tr, i_{n+1})\}$$
of all inputs enabling in $C_2$ after projecting the trace $tr$ on $C_2$.

By construction of $Y$ and $Z$, we have that $Z \subseteq Y$. Since $C_1 \text{ cioco } C_2 \in_c C_2$, then for every $\sigma \in tr \in_c C_2$ and for every $i \in Z$,

$$Out(C_1 \text{ after } (\sigma, i)) \subseteq Out(C_2 \in_c C_2 \text{ after } (\sigma, i)) \quad (1)$$

Let $\Phi = \{\sigma.\langle i'_{n+1} | o'_{n+1} \rangle \mid \sigma \in tr \in_c C_2, o'_{n+1} \in Out(C_1 \text{ after } (\sigma, i'_{n+1})), \text{ and } i_{n+1} \in Z \}$

Since $\sigma.\langle i'_{n+1} | o'_{n+1} \rangle \in tr. \langle i_{n+1} | o_{n+1} \rangle$ and then by the projection definition, one has

$$\pi_i(i'_{n+1}) = i_{n+1} \text{ and } \pi_i(o'_{n+1}) = o_{n+1} \quad (2)$$

By (1), (2) and the definition of $tr. \langle i_{n+1} | o_{n+1} \rangle$ \in_c C_2, we can conclude that $\Phi = tr. \langle i_{n+1} | o_{n+1} \rangle$ \in_c C_2. Thus $tr. \langle i_{n+1} | o_{n+1} \rangle$ \in Trace($\otimes_C C_2$). Consequently, $o_{n+1} \in Out(\otimes_C C_2)$ after $tr. \langle i_{n+1} \rangle$. \hfill \Box

**Proof:** Compositionality for Cartesian product (Theorem 3.2)

Let us assume that

$$C_1 \text{ cioco } \otimes ((C'_1, C'_2)) \in_c 1 \quad \text{ and } \quad C_2 \text{ cioco } \otimes ((C'_1, C'_2)) \in_{c_2}$$

and then prove that $\otimes (C_1, C_2)$ cioco $\otimes ((C'_1, C'_2))$.

Let us use the contradiction principle. For this, let us assume that

$$(\otimes (C_1, C_2)) \text{ cioco } \otimes ((C'_1, C'_2))$$

i.e. there exists a finite trace $tr = \langle (i_1, i'_1) | (o_1, o'_1), \ldots, (i_n, i'_n) | (o_n, o'_n) \rangle \in Trace(\otimes (C'_1, C'_2))$ and $(i, i') \in I_1 \times I_2$ such that there exists an output $(o, o') \in O_1 \times O_2$ among the outputs obtained after executing $(tr, (i, i'))$ on $\otimes (C_1, C_2)$ not belonging to the ones obtained after executing $(tr, (i', i'))$ on $\otimes (C'_1, C'_2)$.

Now, we have

$$tr = \langle (i_1, i'_1) | (o_1, o'_1), \ldots, (i_n, i'_n) | (o_n, o'_n) \rangle \in Trace(\otimes (C_1, C_2))$$

According to the definition of the cartesian product, it is easy to show that the two traces:

$$tr_1 = \langle i_1 | o_1, \ldots, i_n | o_n \rangle \in Trace(C_1)$$

and

$$tr_2 = \langle i'_1 | o'_1, \ldots, i'_n | o'_n \rangle \in Trace(C_2)$$

are respectively the traces involved in $C_1$ and $C_2$ to obtain $tr$.

We also know by the projection definition (see Definition 3.2) that $tr_1 \in Trace(\otimes ((C'_1, C'_2)) \in_1 C_2)$ and $tr_2 \in Trace(\otimes ((C'_1, C'_2)) \in_{c_2} C_2)$.

Since $(o, o') \in Out(\otimes ((C'_1, C'_2)))$ after $(tr, (i, i'))$ and $tr$ is composed of $tr_1$ and $tr_2$, then $o \in Out(C_1)$ after $(tr_1, i)$ and $o' \in Out(C_2)$ after $(tr_2, i')$. Similarly, $o \notin Out(\otimes ((C'_1, C'_2)))$ after $(tr_1, i)$ and $o' \notin Out(\otimes ((C'_1, C'_2)))$ after $(tr_2, i')$ because $(o, o') \notin Out(\otimes ((C'_1, C'_2)))$ after $(tr, (i, i'))$ and $tr_1$ and $tr_2$ are involved to obtain $tr$. Hence, there exists a trace $tr_1 \in Trace(\otimes ((C'_1, C'_2)) \in_{c_1} C_2)$, an input $i$ of $\otimes ((C'_1, C'_2)) \in_{c_1} C_2$ and an output $o \in O_1$ such that $o \in Out(C_1)$ after $(tr_1, i)$ and $o \notin Out(\otimes ((C'_1, C'_2)) \in_{c_1} C_2)$ after $(tr_1, i)$. In the same manner, there exists a trace $tr_2 \in Trace(\otimes ((C'_1, C'_2)) \in_{c_2} C_2)$ an input $i'$ of $\otimes ((C'_1, C'_2)) \in_{c_2} C_2$ and an output $o' \in O_2$ such that $o' \in Out(C_2)$ after $(tr_2, i')$ and $o' \notin Out(\otimes ((C'_1, C'_2)) \in_{c_2} C_2)$ after $(tr_2, i')$. Indeed, this means that $(C_1 \text{ cioco } \otimes ((C'_1, C'_2)))_w$ and $(C_2 \text{ cioco } \otimes ((C'_1, C'_2)))_w$. Hence, we have a contradiction with our hypothesis. \hfill \Box