Enforcing interface flux continuity in enhanced XFEM: stability analysis
Pedro Diez, Sergio Zlotnik, Régis Cottereau

To cite this version:

HAL Id: hal-00784499
https://hal-ecp.archives-ouvertes.fr/hal-00784499
Submitted on 8 Mar 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Enforcing interface flux continuity in enhanced XFEM: stability analysis

PEDRO DÍEZ

(joint work with Sergio Zlotnik and Régis Cottereau)

XFEM is found to be an efficient approach for solving multiphase problems. The model problem reads as follows, find \( u \) taking values in \( \Omega_1 \cup \Omega_2 \) such that

\[
\begin{align*}
\nabla \cdot ( - \nu_1 \nabla u ) &= f & \text{in } \Omega_1 \\
\nabla \cdot ( - \nu_2 \nabla u ) &= f & \text{in } \Omega_2 \\
- \nu \nabla u \cdot \mathbf{n} &= g_N & \text{on } \Gamma_N \\
u_1 \nabla u |_{\Omega_1} \cdot \mathbf{n} &= \nu_2 \nabla u |_{\Omega_2} \cdot \mathbf{n} & \text{on } \Gamma := \partial \Omega_1 \cap \partial \Omega_2
\end{align*}
\]

The level set representation of the phase domains allows having a grid independent of the location of the interface [2]. In order to introduce the necessary gradient discontinuities inside the elements crossed by the interface, XFEM uses the partition of the unity idea to enrich the discretization. In this context, a sensible choice for the enrichment is using a ridge function \( R \) defined as

\[
R = \sum_{i=1}^{n_H} N_i |\phi_i| - \sum_{i=1}^{n_H} N_i \phi_i,
\]

being \( N_i \) the shape functions and \( \phi_i \) the nodal values of the level set, for \( i = 1, \ldots, n_H \), see [1, 3]. Thus, the XFEM approximation reads

\[
u_X = \sum_{i=1}^{n_H} N_i u_i + \sum_{j \in N_a} R N_j a_j,
\]

where the coefficients \( u_i \) for \( i = 1, \ldots, n_H \) are the standard Finite Element nodal unknowns and \( a_j, j \in N_a \), stand for the enriched nodal coefficients.

XFEM provides a much better approximation of the multiphase solution, improving the quality the global quantity (energy like) that the variational form of the problem seeks minimizing. Nevertheless, when applied to diffusion problems in a multiphase setup with high diffusivity contrast, the XFEM strategy suffers from an inaccurate representation of the local fluxes in the vicinity of the interface. The XFEM enrichment improves the global quality of the solution but it is not
properly enforcing any local feature to the fluxes. Thus, the resulting numerical fluxes in the vicinity of the interface are not realistic, in particular when the parametric contrast between the two phases is important. An additional restriction to the XFEM formulation is introduced, aiming at properly reproducing the features of the local fluxes in the transition zone. This restriction is implemented through Lagrange multipliers. The resulting enlarged variational problem reads find the XFEM approximation $u_X \in V_X$ and the (discrete) Lagrange multiplier $\lambda_H \in \tilde{V}_H$ such that

\begin{align}
    a(u_X, w) + b(\lambda_H, w) &= \ell(w) & \forall w \in V_{X,0} \\
    b(\mu, u_X) &= 0 & \forall \mu \in \tilde{V}_H
\end{align}

being $a(\cdot, \cdot)$ the standard bilinear form representing the weak form of problem (1) and

\begin{equation}
    b(\mu, u) := \int_\Gamma (\nu_1 \nabla u|_{\Omega_1} - \nu_2 \nabla u|_{\Omega_2}) \cdot n \mu \, d\Gamma.
\end{equation}

Note that (2b) is the weak form of (1e) and it is the restriction aiming at improving the quality of the flux continuity and, consequently, the quality of the fluxes in the vicinity of the interface. Several examples are presented and the solutions obtained from (2) show a spectacular improvement of the quality of the fluxes with respect to the standard XFEM.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of the semi-hat functions of the Lagrange multipliers space, $\tilde{N}_k$.}
\end{figure}

The problem of choosing the proper Lagrange multiplier space introduces a classical dilemma: if $\tilde{V}_H$ is too small the restriction is not properly enforced and if it is too large the resulting method may be unstable. After some numerical tests, the option selected corresponds to the semi hat functions along the interface, as illustrated in figure 1. In this case, the dimension of $\tilde{V}_H$ is twice the number of elements crossed by the interface.
The mathematical proof of the stability of the numerical scheme requires checking if the LBB condition (also known as inf-sup condition) is fulfilled for the elected spaces and bilinear restriction. We propose a novel approach to prove this proposition by introducing an equivalent form of the theorem and two auxiliary lemmas.

Recall that the well-known LBB compatibility condition, is sufficient to guarantee the stability of the formulation. In other words, the formulation is stable if it exists $k > 0$ such that

$$
\inf_{\mu \in \tilde{V}_H} \sup_{w \in V_X} \frac{b(\mu, w)}{||\mu|| ||w||} \geq k
$$

The LBB condition is equivalent to the following

**Proposition:** $\exists \alpha > 0$ such that $\forall \mu \in \tilde{V}_H$, $\exists v \in V_X$ verifying

$$
(5a) \quad ||\nu \nabla \cdot n|| = \mu
$$

$$
(5b) \quad ||v||_{V_X} \leq \alpha ||\mu||_{\tilde{V}_H}
$$

The equivalence is straightforwardly shown by considering that

$$
b(\mu, w) = \int_{\Gamma} \mu^2 d\Gamma = ||\mu|| \text{ and } \frac{b(\mu, v)}{||\mu|| ||v||} = \frac{||\mu||}{||v||}
$$

Thus, since $||v|| \leq \alpha ||\mu||$, taking $k = 1/\alpha$ the LBB condition follows.

The latter proposition is reduced to proof the two following lemmas.

**Lemma 1 (local version of the proposition, restricted to one element):** Let $\Omega^k$ be one linear triangular element crossed by the interface $\Gamma$. The restriction of $\Gamma$ to $\Omega^k$ is denoted $\Gamma^k$. The nodes of $\Omega^k$ are denoted $P_1$, $P_2$ and $P_3$, choosing the order such that $P_1$ and $P_2$ are on the same side of the interface. As classically done in XFEM, we assume that $\exists \epsilon > 0$ such that $|\Gamma^k| > \epsilon$. The restrictions of the functional spaces $V_X$ and $\tilde{V}_H$ to $\Omega^k$ and $\Gamma^k$ are denoted $V_X^k$ and $\tilde{V}_H^k$, with respective norms $||v||_{V_X}^k = \int_{\Omega^k} v^2 d\Omega$ and $||\mu||_{\tilde{V}_H}^k = \int_{\Gamma^k} \mu^2 d\Gamma$. The standard FE shape function corresponding to the node $P_1$ is denoted $N_1$, and the ridge function $R$.

Then, $\exists \alpha > 0$ such that $\forall \mu \in \tilde{V}_H^k$, $\exists v \in \text{span}\{N_1, RN_1\} \subset V_X^k$ (i.e. describing $v$ with the d.o.f. corresponding to $P_1$ only) verifying

$$
(6a) \quad ||\nu \nabla \cdot n|| = \mu
$$

$$
(6b) \quad ||v||_{V_X^k} \leq \alpha ||\mu||_{\tilde{V}_H^k}
$$

**Lemma 2 (controlled propagation of the norm along the interface elements strip):** Let $\Omega^k$ and $\Omega^{k+1}$ be two contiguous elements crossed by the interface. Let us denote $P_1$ and $P_3$ the common nodes to $\Omega^k$ and $\Omega^{k+1}$, being $P_2$ the third node in $\Omega^k$. $P_1$ is selected such that it is on the same side of the interface as $P_2$. The third node in $\Omega^{k+1}$ is denoted as $P_4$. Then, $\exists \beta > 0$ such that, for any $v$ defined by the d.o.f. of $\Omega^k$, $v \in \text{span}\{N_i, RN_i\}$, $i = 1, 2, 3$ it holds that $||v||_{V_X^{k+1}} \leq \beta ||v||_{V_X^k}$. 
Isoparametric $C_0$ Interior Penalty Methods for Plate Bending Problems on Smooth Domains

Li-yeng Sung (joint work with Susanne C. Brenner and Michael Neilan)

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$ such that $\partial \Omega$ is the union of the disjoint closed curves $\Gamma_C$, $\Gamma_S$, and $\Gamma_F$. The bending problem of a thin Kirchhoff plate [7] is to find $u \in V$ such that

$$
\int_{\Omega} \left\{ \left( \Delta w \right) \left( \Delta v \right) - (1 - \nu) [w, v] \right\} \, dx = \int_{\Omega} fv \, dx \quad \forall \ v \in V,
$$

where $V = \{ v \in H^2(\Omega): v = 0 \text{ on } \Gamma_C \cup \Gamma_S \text{ and } \partial v / \partial n = 0 \text{ on } \Gamma_C \}$, $\nu \in (0, 1/2)$ is the Poisson ratio, and $[w, v] = (\partial^2 w / \partial x_1^2)(\partial^2 v / \partial x_2^2) + (\partial^2 w / \partial x_2^2)(\partial^2 v / \partial x_1^2) - 2(\partial^2 w / \partial x_1 \partial x_2)(\partial^2 v / \partial x_1 \partial x_2)$ is the Monge-Ampère bilinear form. Here $u$ is the vertical displacement of the middle surface of the plate, $f \in L^2(\Omega)$ is the vertical load density divided by the flexural rigidity of the plate, and $\Gamma_C$ (resp. $\Gamma_S$ and $\Gamma_F$) is the clamped (resp. simply supported and free) part of $\partial \Omega$.

We assume $|\Gamma_C| + |\Gamma_S| > 0$ so that (1) is well-posed. In order to obtain high order convergence, the computational domain of a finite element method for (1) must approximate $\Omega$ to a high order. For second order problems this can be accomplished by the isoparametric approach [6, 9, 4]. But the construction of $C_1$ finite element methods on non-polygonal domains is much more complicated [14, 15, 12, 10, 11, 13].

We have shown in [3] that by combining isoparametric finite element spaces for second order problems with the $C_0$ interior penalty methodology [8, 5, 2], it is possible to solve the plate bending problem (1) efficiently. In this approach the discrete problem is obtained by the following procedure.

We construct a curvilinear polygon $\Omega_h$ whose corners belong to $\partial \Omega$ such that $\Omega_h$ is the union of the triangles in a quasi-uniform triangulation $T_h$. We assume that $\Omega_h$ is a good approximation of $\Omega$ so that $\partial \Omega_h$ is the union of the disjoint closed subsets $\Gamma_{C,h}$, $\Gamma_{S,h}$, and $\Gamma_{F,h}$ that approximate $\Gamma_C$, $\Gamma_S$ and $\Gamma_F$ respectively. Each of the triangles in $T_h$ can have at most one curved edge and only those triangles with more than one vertex on $\partial \Omega$ can have a curved edge. We assume that each triangle...